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Maximal atmospheric neutrino mixing and the small ratio of muon to tau mass

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Abstract

We discuss the problem of the small ratio of muon mass to tau mass in a class of seesaw models where maximal atmospheric neutrino mixing is enforced through a μ–τ interchange symmetry. We introduce into those models an additional symmetry K such that $m_\mu = 0$ in the case of exact K invariance. The symmetry K may be softly broken in the Higgs potential, and one thus achieves $m_\mu \ll m_\tau$ in a technically natural way. We speculate on a wider applicability of this mechanism.

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1 Introduction

Lepton mixing seems to be an established fact now—for reviews see, for instance, Ref. [1]. The solar neutrino mixing has turned out to be large but non-maximal, whereas it is likely that the atmospheric mixing angle is maximal, i.e. $\pi/4$ or close to that value.

On the theoretical side, a popular way of generating small neutrino masses is the seesaw mechanism [2]. It has three sources of lepton mixing: the charged-lepton mass matrix M_ℓ, the Dirac mass matrix M_D linking the left-handed neutrinos ν_L to the right-handed neutrinos ν_R, and the Majorana mass matrix M_R of the right-handed neutrinos; the mass Lagrangian is

$$\mathcal{L}_{\mathrm{mass}} = -\bar{\ell}_R M_\ell \ell_L - \bar{\nu}_R M_D \nu_L - \frac{1}{2}\,\bar{\nu}_R M_R C \bar{\nu}_R^T + \mathrm{H.c.} \tag{1}$$

The mass matrix of the light neutrinos is then given by

$$\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \,. \tag{2}$$

Some models have been proposed in the literature [3, 4, 5, 6] which account for maximal atmospheric neutrino mixing by invoking non-abelian symmetries such that M_ℓ and M_D are simultaneously diagonal. In this case the mass matrix M_R is the sole source of neutrino mixing. An interesting common feature of the models of Refs. [4, 5, 6] is that they need a minimum of three Higgs doublets in order to have enough freedom to accommodate all lepton masses.

In this paper we investigate the "$\mathbb{Z}_2$ model" of Ref. [4], the "D_4 model" of Ref. [5], and the "CP model" of Ref. [6]. The models of Refs. [4, 5] have non-abelian horizontal-symmetry groups. The model of Ref. [6] has a non-standard CP symmetry [7]—for the general idea of non-standard CP transformations see Ref. [8]—instead of a certain $\mathbb{Z}_2$ contained in the symmetry groups of Refs. [4, 5]. The symmetries defining the models of Refs. [4, 5, 6] will be described in the respective sections of this paper; there, their scalar sectors are also discussed.

In all those models the masses of the charged leptons are free and must be adjusted by finetuning. The finetuning problem of adjusting the small ratio of muon mass (m_μ) to tau mass (m_τ) is more complex than in the Standard Model, as we shall see below. In Ref. [5] we have proposed a solution to this problem by introducing an additional symmetry operation K—but no additional fields—which restricts the coupling constants of the model in such a way that $m_\mu = 0$ with exact K invariance.[1] Through soft K breaking a non-zero ratio m_μ/m_τ is then achieved, and this solves the finetuning problem in a technically natural way. The problem of reconciling maximal atmospheric neutrino mixing with a small ratio m_μ/m_τ was also addressed in Ref. [9] in a class of models different from the ones discussed here.

The purpose of this paper is twofold: firstly, we present the detailed proof of the existence of this mechanism and work out the principles on which it is based; secondly,

[1]In the paper of Ref. [5] we have used T instead of K for denoting the extra symmetry which leads to $m_\mu = 0$. Since T is commonly used in physics for the time-reversal symmetry, in this paper and in Ref. [6] we have switched the notation to K.

we show that it may operate not only in the model of Ref. [5] but also in those of Refs. [4, 6]. It will become evident to which class of models the symmetry K can be applied.

In Section 2 we describe the multiplets and Yukawa Lagrangians of the models of Refs. [4, 5, 6], we point out the finetuning problem for $m_\mu \ll m_\tau$, and we introduce the symmetry K, which reduces the finetuning to a problem of achieving nearly equal vacuum expectation values (VEVs) for two Higgs doublets. The latter problem is addressed by investigating the effect of K on the Higgs potentials of the $\mathbb{Z}_2$, D_4, and CP models in Sections 3, 4, and 5, respectively. Section 6 contains a summary.

2 The Yukawa Lagrangians and K

In Refs. [4, 5] there are the following Yukawa couplings in the lepton sector:

$$\begin{aligned} \mathcal{L}_\mathrm{Y} \; = \; & -y_1 \bar{D}_e \nu_{eR} \tilde{\phi}_1 - y_2 \left(\bar{D}_\mu \nu_{\mu R} + \bar{D}_\tau \nu_{\tau R} \right) \tilde{\phi}_1 \\ & -y_3 \bar{D}_e e_R \phi_1 - y_4 \left(\bar{D}_\mu \mu_R + \bar{D}_\tau \tau_R \right) \phi_2 - y_5 \left(\bar{D}_\mu \mu_R - \bar{D}_\tau \tau_R \right) \phi_3 + \mathrm{H.c.} \end{aligned} \tag{3}$$

The constants y_i $(i = 1, \ldots, 5)$ are in general complex. Denoting the VEV of ϕ_j^0 $(j = 1, 2, 3)$ by $v_j \big/ \sqrt{2}$, the muon and tau masses are given by

$$m_\mu = \frac{1}{\sqrt{2}} \left| y_4 v_2 + y_5 v_3 \right|, \quad m_\tau = \frac{1}{\sqrt{2}} \left| y_4 v_2 - y_5 v_3 \right|, \tag{4}$$

respectively.

A variant of the Lagrangian of Eq. (3), namely

$$\begin{aligned} \mathcal{L}_\mathrm{Y}^\prime \; = \; & -y_1 \bar{D}_e \nu_{eR} \tilde{\phi}_1 - \left(y_2 \bar{D}_\mu \nu_{\mu R} + y_2^* \bar{D}_\tau \nu_{\tau R} \right) \tilde{\phi}_1 \\ & -y_3 \bar{D}_e e_R \phi_1 - \left(y_4 \bar{D}_\mu \mu_R + y_4^* \bar{D}_\tau \tau_R \right) \phi_2 - \left(y_5 \bar{D}_\mu \mu_R - y_5^* \bar{D}_\tau \tau_R \right) \phi_3 + \mathrm{H.c.}\,, \end{aligned} \tag{5}$$

is found in the model of Ref. [6]. Here, y_1 and y_3 are real, but the remaining y_i are in general complex; the muon and tau masses are given by

$$m_\mu = \frac{1}{\sqrt{2}} \left| y_4 v_2 + y_5 v_3 \right|, \quad m_\tau = \frac{1}{\sqrt{2}} \left| y_4^* v_2 - y_5^* v_3 \right|, \tag{6}$$

respectively.

In all three models under discussion there is an "auxiliary" symmetry of the $\mathbb{Z}_2$ type:

$$\mathbb{Z}_2^{(\mathrm{aux})}: \; \nu_{eR},\ \nu_{\mu R},\ \nu_{\tau R},\ \phi_1,\ e_R \text{ change sign.} \tag{7}$$

It is spontaneously broken by v_1 and it restricts the couplings of the Higgs doublets ϕ_j as seen in the Yukawa Lagrangians of Eqs. (3) and (5).

One must use finetuning for obtaining $m_\mu \ll m_\tau$. In the case of the Lagrangians of Eqs. (3) and (5) one does not simply have to choose one Yukawa coupling to be small, like in the Standard Model, rather one has to choose two products of unrelated quantities—one Yukawa coupling and one VEV—such that in m_μ those two products nearly cancel.

In order to soften the amount of finetuning we have proposed in Ref. [5] to add to the model of that paper the symmetry

$$K : \quad \mu_R \to -\mu_R\,, \quad \phi_2 \leftrightarrow \phi_3\,. \tag{8}$$

That symmetry leads to

$$y_4 = -y_5 \tag{9}$$

in both the Lagrangians of Eqs. (3) and (5). In this way the finetuning is confined to the VEVs:

$$\frac{m_\mu}{m_\tau} = \left| \frac{v_2 - v_3}{v_2 + v_3} \right|. \tag{10}$$

The symmetry K also has effects upon the scalar potential V, which we now turn to investigate. In this task, there are some differences among the scalar sectors and among the symmetries of the models of Refs. [4, 5, 6] which must be taken into consideration.

3 The $\mathbb{Z}_2$ model

The scalar sector of the model of Ref. [4] consists solely of the three Higgs doublets ϕ_j. Its symmetries are the following:

- the three groups $U(1)_{L_\alpha}$ ($\alpha = e, \mu, \tau$) associated with the lepton numbers L_α, which are softly broken by the Majorana mass terms of the right-handed neutrino singlets, i.e. by the matrix M_R;
- a $\mathbb{Z}_2$-type symmetry given by

$$\mathbb{Z}_2^{(\mathrm{tr})} : \; D_\mu \leftrightarrow D_\tau\,, \; \mu_R \leftrightarrow \tau_R\,, \; \nu_{\mu R} \leftrightarrow \nu_{\tau R}\,, \; \phi_3 \to -\phi_3\,, \tag{11}$$

  which transposes the second and third families and is spontaneously broken by v_3;
- the symmetry $\mathbb{Z}_2^{(\mathrm{aux})}$ of Eq. (7).

Because of the two $\mathbb{Z}_2$-type symmetries the Higgs potential V of this model is invariant under the three independent sign changes

$$\phi_j \to -\phi_j\,, \quad \phi_{j'} \to \phi_{j'} \quad (j' \neq j)\,, \tag{12}$$

where $j, j' = 1, 2, 3$. Therefore, in every term of V each Higgs doublet can occur only either two or four times.

We define the Higgs potential V_ϕ as the polynomial of order 4 in the ϕ_j obeying the symmetries of Eqs. (8) and (12). *Soft breaking* of the symmetry K of Eq. (8) is achieved by adding to V_ϕ the unique term

$$V_{\mathrm{soft}} = \mu_{\mathrm{soft}} \left(\phi_2^\dagger \phi_2 - \phi_3^\dagger \phi_3 \right). \tag{13}$$

We want to show the following [5]:

(a) Under certain conditions on the coupling constants, the minimum of V_ϕ fulfills

$$v_2 = v_3 \,. \tag{14}$$

Equation (10) then gives $m_\mu = 0$.

(b) The full potential $V = V_\phi + V_{\mathrm{soft}}$ leads to a non-vanishing m_μ. For small μ_{soft} the ratio m_μ / m_τ is small in a technically natural way.

Point (a): V_ϕ is given by

$$\begin{aligned} V_\phi &= -\mu_1 \phi_1^\dagger \phi_1 - \mu_2 \left(\phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3 \right) \\ &\quad + \lambda_1 \left(\phi_1^\dagger \phi_1 \right)^2 + \lambda_2 \left[\left(\phi_2^\dagger \phi_2 \right)^2 + \left(\phi_3^\dagger \phi_3 \right)^2 \right] \\ &\quad + \lambda_3 \left(\phi_1^\dagger \phi_1 \right) \left(\phi_2^\dagger \phi_2 + \phi_3^\dagger \phi_3 \right) + \lambda_4 \left(\phi_2^\dagger \phi_2 \right) \left(\phi_3^\dagger \phi_3 \right) \\ &\quad + \lambda_5 \left[\left(\phi_1^\dagger \phi_2 \right) \left(\phi_2^\dagger \phi_1 \right) + \left(\phi_1^\dagger \phi_3 \right) \left(\phi_3^\dagger \phi_1 \right) \right] + \lambda_6 \left(\phi_2^\dagger \phi_3 \right) \left(\phi_3^\dagger \phi_2 \right) \\ &\quad + \lambda_7 \left[\left(\phi_2^\dagger \phi_3 \right)^2 + \left(\phi_3^\dagger \phi_2 \right)^2 \right] + \lambda_8 \left[\left(\phi_1^\dagger \phi_2 \right)^2 + \left(\phi_1^\dagger \phi_3 \right)^2 \right] + \lambda_8^* \left[\left(\phi_2^\dagger \phi_1 \right)^2 + \left(\phi_3^\dagger \phi_1 \right)^2 \right] . \end{aligned} \tag{15}$$

All the coupling constants in V_ϕ, except for λ_8, are real. We replace the Higgs doublets in V_ϕ by their VEVs, parameterized as

$$\frac{v_1}{\sqrt{2}} = u_1 \,, \quad \frac{v_2}{\sqrt{2}} = u e^{i\alpha} \cos\sigma \,, \quad \frac{v_3}{\sqrt{2}} = u e^{i\beta} \sin\sigma \,, \tag{16}$$

where, without loss of generality, u_1 and u are positive and σ belongs to the first quadrant. Note that $\sqrt{2 \left(u_1^2 + u^2 \right)} \simeq 246\,\mathrm{GeV}$ represents the electroweak scale. In this way we obtain the function

$$\begin{aligned} F_\phi \equiv \langle 0 \left| V_\phi \right| 0 \rangle &= -\mu_1 u_1^2 - \mu_2 u^2 + \lambda_1 u_1^4 + \lambda_2 u^4 + \left(\lambda_3 + \lambda_5 \right) u_1^2 u^2 \\ &\quad + \left[\tilde{\lambda} - 4 \lambda_7 \sin^2 \left(\alpha - \beta \right) \right] u^4 \cos^2\sigma \sin^2\sigma \\ &\quad + 2 \left| \lambda_8 \right| u_1^2 u^2 \left[\cos^2\sigma \, \cos \left(\epsilon + 2\alpha \right) + \sin^2\sigma \, \cos \left(\epsilon + 2\beta \right) \right] , \end{aligned} \tag{17}$$

where $\tilde{\lambda} \equiv -2\lambda_2 + \lambda_4 + \lambda_6 + 2\lambda_7$ and $\epsilon \equiv \arg \lambda_8$. In searching for the minimum of F_ϕ in terms of σ, α, and β we make the following simple observation: if

$$\tilde{\lambda} < 0 \quad \text{and} \quad \lambda_7 < 0 \,, \tag{18}$$

then the minimum of F_ϕ has

$$\sigma = \frac{\pi}{4} \,, \quad \alpha = \beta = \frac{\pi - \epsilon}{2} \,, \tag{19}$$

i.e.

$$v_2 = v_3 = u e^{i (\pi - \epsilon)/2} \,. \tag{20}$$

With Eq. (20) one obtains $m_\mu = 0$, which is close to the real situation if one takes into account that $m_\mu \ll m_\tau$. Equation (18) formulates sufficient conditions for the minimum of V_ϕ to obey the relation (20).

Point (b): Now we take into account V_{soft}, which gives

$$\langle 0|V_{\mathrm{soft}}|0\rangle = \mu_{\mathrm{soft}} u^2 \cos 2\sigma\,. \tag{21}$$

We must minimize $\langle 0\,|V_\phi + V_{\mathrm{soft}}|\,0\rangle$. Since the new term in Eq. (21) does not contain the phases α and β, the minimum is again at $\alpha = \beta = (\pi - \epsilon)/2$. Then, the second line of Eq. (17) together with the term stemming from V_{soft} lead to

$$\cos 2\sigma = \frac{2\mu_{\mathrm{soft}}}{\tilde\lambda u^2}\,. \tag{22}$$

It is obvious that, if $|\mu_{\mathrm{soft}}|$ is sufficiently smaller than u^2, the angle σ will still be close to $\pi/4$. Using Eqs. (10) and (16), we obtain

$$\frac{m_\mu}{m_\tau} = \frac{|\cos 2\sigma|}{1+\sqrt{1-\cos^2 2\sigma}} \simeq \left|\frac{\mu_{\mathrm{soft}}}{\tilde\lambda u^2}\right|, \tag{23}$$

showing explicitly that a small μ_{soft} leads to a small m_μ/m_τ.

Let us elaborate on the magnitude of μ_{soft}. The Higgs-potential coupling $\tilde\lambda$ could be of order 0.1. Furthermore, let us choose, for instance, $u = 172$ GeV. Then it turns out that $u_1 \simeq 26$ GeV, which is much smaller than u; this is reasonable since u_1 is responsible both for the electron mass and for the neutrino masses. From Eq. (23) one then obtains $|\mu_{\mathrm{soft}}| \sim 176\ \mathrm{GeV}^2$. Thus, $|\mu_{\mathrm{soft}}|$ is much smaller than u^2, yet it may well be of the order of magnitude or even larger than u_1^2. On the other hand, if u_1 is comparable to u, then we shall have $|\mu_{\mathrm{soft}}|$ much smaller than both u^2 and u_1^2, but its order of magnitude will not change with respect to the numerical example above.

4 The D_4 model

The model of Ref. [5] is based on the horizontal-symmetry group D_4. It has the same gauge multiplets of the model of the previous section, plus two real scalar gauge singlets χ_k $(k = 1, 2)$. The symmetries are the following: the horizontal group D_4 is generated by the $\mathbb{Z}_2^{(\mathrm{tr})}$ of Eq. (11), supplemented by $\chi_1 \leftrightarrow \chi_2$, and by an additional $\mathbb{Z}_2$-type symmetry

$$\mathbb{Z}_2^{(\tau)}:\ D_\tau,\ \tau_R,\ \nu_{\tau R},\ \chi_2 \text{ change sign.} \tag{24}$$

Besides D_4, there is also the symmetry $\mathbb{Z}_2^{(\mathrm{aux})}$ of Eq. (7). Both D_4 and $\mathbb{Z}_2^{(\mathrm{aux})}$ are spontaneously broken. We furthermore impose the symmetry K of Eq. (8).

The full potential has the structure

$$V = V_\chi + V_{\chi\phi} + V_\phi + V_{\mathrm{soft}}\,, \tag{25}$$

where V_χ contains only the fields χ_k whereas the terms in $V_{\chi\phi}$ contain both the χ_k and the ϕ_j. Because $\mathbb{Z}_2^{(\mathrm{aux})}$ and $\mathbb{Z}_2^{(\mathrm{tr})}$ hold in this case as well, V_ϕ and V_{soft} are the same as in the previous section. We will show that—like for V_ϕ under point (a) in the previous section—there is a minimum of $V_\chi + V_{\chi\phi} + V_\phi$ for which Eq. (14) holds. Then the procedure of point (b) works in the present model as well.

The potentials V_χ and $V_{\chi\phi}$ are given by

$$
\begin{aligned}
V_\chi &= -\mu\left(\chi_1^2+\chi_2^2\right)+\lambda\left(\chi_1^2+\chi_2^2\right)^2+\lambda'\left(\chi_1^2-\chi_2^2\right)^2, && (26)\\
V_{\chi\phi} &= \left(\chi_1^2+\chi_2^2\right)\left[\rho_1\phi_1^\dagger\phi_1+\rho_2\left(\phi_2^\dagger\phi_2+\phi_3^\dagger\phi_3\right)\right]+\eta\left(\chi_1^2-\chi_2^2\right)\left(\phi_2^\dagger\phi_3+\phi_3^\dagger\phi_2\right). && (27)
\end{aligned}
$$

In Eqs. (26) and (27) all the coupling constants are real. We parameterize the VEVs of the χ_k in the following way [5]:

$$
\langle 0\left|\chi_1\right|0\rangle = W\cos\gamma\,, \quad \langle 0\left|\chi_2\right|0\rangle = W\sin\gamma\,. \tag{28}
$$

We then obtain

$$
\begin{aligned}
\langle 0\left|V_\chi+V_{\chi\phi}\right|0\rangle &= -\mu W^2+\lambda W^4+W^2\left(\rho_1u_1^2+\rho_2u^2\right)\\
&\quad +2\eta W^2u^2\cos 2\gamma\cos\sigma\sin\sigma\cos\left(\alpha-\beta\right)+\lambda' W^4\cos^2 2\gamma\,. && (29)
\end{aligned}
$$

The minimum of γ is found at

$$
\cos 2\gamma = -\frac{\eta u^2\cos\sigma\sin\sigma\cos\left(\alpha-\beta\right)}{\lambda' W^2}\,, \tag{30}
$$

provided $\lambda' > 0$ and the quantity in the right-hand side of Eq. (30) is smaller than unity in modulus.[2] Inserting the result of Eq. (30) into Eq. (29) we arrive at

$$
\begin{aligned}
\langle 0\left|V_\chi+V_{\chi\phi}\right|0\rangle &= -\mu W^2+\lambda W^4+W^2\left(\rho_1u_1^2+\rho_2u^2\right)\\
&\quad -\frac{\eta^2}{\lambda'}\,u^4\cos^2\sigma\sin^2\sigma\cos^2\left(\alpha-\beta\right). && (31)
\end{aligned}
$$

Adding this expression to F_ϕ in Eq. (17), we see that this amounts to the replacements $\tilde\lambda \to \tilde\lambda - \eta^2/\lambda'$ and $4\lambda_7 \to 4\lambda_7 - \eta^2/\lambda'$ in Eq. (17). Then the minimum of $V_\chi + V_{\chi\phi} + V_\phi$ with respect to σ, α, and β is not changed compared to Eq. (19); we have again $v_2 = v_3$ and, therefore, $m_\mu = 0$. As before, breaking the symmetry K softly through V_soft one achieves a non-zero muon mass. The muon mass is small in a technically natural way by requiring only one parameter, μ_soft, to be small.

5 The CP model

The model of Ref. [6] has the same scalar multiplets and nearly the same symmetries as the $\mathbb{Z}_2$ model of Section 3; the only difference is that the $\mathbb{Z}_2^{(\mathrm{tr})}$ of Eq. (11) is replaced by the non-standard CP symmetry

$$
\begin{array}{rcl}
D_\alpha & \to & iS_{\alpha\beta}\gamma^0 C\bar D_\beta^T\,,\\
\alpha_R & \to & iS_{\alpha\beta}\gamma^0 C\bar \beta_R^T\,,\\
\nu_{\alpha R} & \to & iS_{\alpha\beta}\gamma^0 C\bar \nu_{\beta R}^T\,,\\
\phi_{1,2} & \to & \phi_{1,2}^*\,,\\
\phi_3 & \to & -\phi_3^*\,,
\end{array}
\quad \text{with} \quad S = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}. \tag{32}
$$

[2]In the model of Ref. [5] we need $W \gg u$ anyway.

Here, $\alpha, \beta = e, \mu, \tau$.

This replacement not only leads to the Yukawa Lagrangian of Eq. (5), slightly different from the one of Eq. (3), but also allows for a more general Higgs potential, due to the complex conjugation in the transformation of the Higgs doublets in Eq. (32). One can show that invariance under the non-standard CP transformation and under the horizontal symmetry K allows, in addition to the terms in V_ϕ of Eq. (15), two more terms:

$$\begin{aligned} V_9 &= i\lambda_9 \left[\left(\phi_1^\dagger \phi_2 \right) \left(\phi_1^\dagger \phi_3 \right) - \left(\phi_2^\dagger \phi_1 \right) \left(\phi_3^\dagger \phi_1 \right) \right], && (33) \\ V_{10} &= i\lambda_{10} \left(\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_2 \right) \left(\phi_2^\dagger \phi_2 - \phi_3^\dagger \phi_3 \right). && (34) \end{aligned}$$

The coupling constants λ_9 and λ_{10} are real, and now the same holds for λ_8 in Eq. (15), i.e. $\epsilon \equiv \arg \lambda_8 = 0$ or π. As for the soft breaking of K, besides V_{soft}, the CP-invariant term

$$V'_{\mathrm{soft}} = i\mu'_{\mathrm{soft}} \left(\phi_2^\dagger \phi_3 - \phi_3^\dagger \phi_2 \right) \quad (35)$$

is now allowed too. Thus the full potential is $V = V_\phi + V_9 + V_{10} + V_{\mathrm{soft}} + V'_{\mathrm{soft}}$. The VEVs of the additional terms in the potential are given by

$$\begin{aligned} \langle 0 \left| V_9 \right| 0 \rangle &= -\lambda_9 u_1^2 u^2 \sin 2\sigma \sin(\alpha + \beta), && (36) \\ \langle 0 \left| V_{10} \right| 0 \rangle &= \lambda_{10} u^4 \cos 2\sigma \sin 2\sigma \sin(\alpha - \beta), && (37) \\ \langle 0 \left| V'_{\mathrm{soft}} \right| 0 \rangle &= \mu'_{\mathrm{soft}} u^2 \sin 2\sigma \sin(\alpha - \beta). && (38) \end{aligned}$$

To find the minimum of the potential, we proceed in analogy to Section 3. We define a function $\mathcal{F}_\phi = F_\phi + \langle 0 \left| V_9 \right| 0 \rangle + \langle 0 \left| V_{10} \right| 0 \rangle$ and search for its minimum with respect to σ, α, and β. We find

$$\sigma = \frac{\pi}{4}, \quad \alpha = \beta = \omega, \quad (39)$$

which agrees with the minimum of V_ϕ in Eq. (19), except that the value of ω is now determined by

$$2\lambda_8 \sin 2\omega + \lambda_9 \cos 2\omega = 0. \quad (40)$$

Switching on the two soft K-breaking terms, the minimum in Eq. (39) gets shifted. Because of the complication with the three additional terms in V, we are unable to give an exact solution for the shifted minimum, like we did in Section 3. We therefore resort to a discussion of the shift of the minimum to first order in the small quantities μ_{soft} and μ'_{soft}. We define δ_0, δ_+, and δ_- as characterizing the deviations of σ, α, and β from their values in Eq. (39):

$$\sigma = \frac{\pi}{4} - \frac{\delta_0}{2}, \quad \alpha = \omega + \delta_+ + \frac{\delta_-}{2}, \quad \beta = \omega + \delta_+ - \frac{\delta_-}{2}. \quad (41)$$

We expand the function $\mathcal{F}_\phi$ to second order, and the soft-breaking terms to first order, in these small variables. Dropping the constant terms, we obtain the expansion

$$\frac{1}{2} \sum_{a,b} \mathcal{F}_{ab} \delta_a \delta_b + \sum_a f_a \delta_a, \quad (42)$$

where $\mathcal{F} \equiv (\mathcal{F}_{ab})$ is the symmetric and positive matrix of the second derivatives of $\mathcal{F}_\phi$ at the minimum (39):

$$\begin{array}{rcl}
\mathcal{F}_{00} & = & -\frac{1}{2}\,\tilde{\lambda} u^4 + \lambda_9 u_1^2 u^2 \sin 2\omega\,, \\
\mathcal{F}_{++} & = & \left(-8\lambda_8 \cos 2\omega + 4\lambda_9 \sin 2\omega\right) u_1^2 u^2\,, \\
\mathcal{F}_{--} & = & -2\lambda_7 u^4 - 2\lambda_8 u_1^2 u^2 \cos 2\omega\,, \\
\mathcal{F}_{0-} & = & -2\lambda_8 u_1^2 u^2 \sin 2\omega + \lambda_{10} u^4\,, \\
\mathcal{F}_{0+} & = & 0\,, \\
\mathcal{F}_{+-} & = & 0\,.
\end{array} \tag{43}$$

The second term in Eq. (42) represents the expansion of the soft-breaking terms, with

$$f_0 = \mu_{\mathrm{soft}} u^2\,, \quad f_+ = 0\,, \quad f_- = \mu'_{\mathrm{soft}} u^2\,. \tag{44}$$

Let us now define the vectors $\mathbf{f} = (f_0, f_+, f_-)^T$ and $\boldsymbol{\delta} = (\delta_0, \delta_+, \delta_-)^T$. The minimum of V is given, to first order in the δ_a and in the soft-breaking parameters, by

$$\boldsymbol{\delta} = -\mathcal{F}^{-1}\mathbf{f}\,. \tag{45}$$

Using Eqs. (45), (43), and (44) it is straightforward to calculate the shifts δ_a. From the zeros in $\mathcal{F}$ and $\mathbf{f}$ one immediately concludes that $\delta_+ = 0$, i.e. the quantity $\alpha + \beta$ remains 2ω to first order in the small ratios $\mu_{\mathrm{soft}}/u_1^2$, $\mu'_{\mathrm{soft}}/u_1^2$, μ_{soft}/u^2, and μ'_{soft}/u^2. The ratio of the muon mass to the tau mass is expressed as

$$\frac{m_\mu}{m_\tau} = \left|\frac{v_2 - v_3}{v_2 + v_3}\right| \simeq \frac{1}{2}\left|\delta_0 + i\delta_-\right|\,, \tag{46}$$

showing that a non-zero muon mass is generated not only by the different absolute values of v_2 and v_3, as in Section 3, but also by the different phases of those VEVs.

6 Summary

In this paper we have shown that the models of Refs. [4, 5, 6], which achieve maximal atmospheric neutrino mixing through non-abelian symmetries, allow for a technically natural explanation of the smallness of the muon mass as compared to the tau mass.

The models are characterized by diagonal Yukawa couplings, as seen in the Lagrangian of Eq. (3), valid for the models of Refs. [4, 5], and in the Lagrangian of Eq. (5), in the case of Ref. [6]. At face value those Yukawa Lagrangians suggest that m_μ and m_τ should be of the same order of magnitude. In this paper, the key for $m_\mu \ll m_\tau$ was the extra horizontal symmetry K of Eq. (8). As a consequence of that symmetry the ratio m_μ/m_τ is given by Eq. (10), which is a function only of the VEVs v_2 and v_3. We have shown that the invariance of the Higgs potentials under K and under the additional symmetries of the various models is such that they all admit minima with $v_2 = v_3$, leading to $m_\mu = 0$. We then break K softly by the term in Eq. (13), which is unique in the cases of the $\mathbb{Z}_2$ model of Ref. [4] and of the D_4 model of Ref. [5]; in the case of the CP model of Ref. [6] there is the additional K soft-breaking term of Eq. (35). In this way, we link the smallness of m_μ/m_τ to the smallness of the soft-breaking terms.

As for the electron mass, it can be read off from the Yukawa Lagrangians (3) and (5) that the smallness of m_e is linked to the smallness of the neutrino masses through the small VEV of ϕ_1^0.

We feel that seesaw models where neutrino mixing stems solely from the Majorana mass matrix of the heavy neutrino singlets are quite appealing, since they allow for symmetries that enforce maximal atmospheric neutrino mixing. It is, therefore, possible that our mechanism for $m_\mu \ll m_\tau$, which is connected with Yukawa Lagrangians of the type in Eqs. (3) and (5) and does not need any new fields, has a wider applicability and is not confined to the models of Refs. [4, 5, 6] discussed in this paper.

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References

[1] W. Grimus, *Neutrino physics—Theory*, lectures given at the *41. Internationale Universitätswochen für Theoretische Physik, Flavour Physics*, Schladming, Styria, Austria, 22–28 February 2003 [hep-ph/0307149];
J.W.F. Valle, *Neutrino masses twenty-five years later*, invited talk presented at MRST'03, Syracuse, New York, May 2003 [hep-ph/0307192];
V. Barger, D. Marfatia, and K. Whisnant, *Progress in the physics of massive neutrinos*, hep-ph/0308123.

[2] M. Gell-Mann, P. Ramond, and R. Slansky, *Complex spinors and unified theories*, in *Supergravity, Proceedings of the Workshop, Stony Brook, New York, 1979*, eds. P. van Nieuwenhuizen and D.Z. Freedman (North Holland, Amsterdam, 1979);
T. Yanagida, *Horizontal gauge symmetry and masses of neutrinos*, in *Proceedings of the Workshop on Unified Theories and Baryon Number in the Universe*, Tsukuba, Japan, 1979, eds. O. Sawada and A. Sugamoto (KEK report no. 79–18, Tsukuba, 1979);
R.N. Mohapatra and G. Senjanović, *Neutrino mass and spontaneous parity violation*, Phys. Rev. Lett. 44 (1980) 912.

[3] K.S. Babu, E. Ma, and J.W.F. Valle, *Underlying A_4 symmetry for the neutrino mass matrix and the quark mixing matrix*, Phys. Lett. B 552 (2003) 207 [hep-ph/0206292]; see also E. Ma, *Plato's fire and the neutrino mass matrix*, Mod. Phys. Lett. A 17 (2002) 2361 [hep-ph/0211393].

[4] W. Grimus and L. Lavoura, *Softly broken lepton numbers and maximal neutrino mixing*, J. High Energy Phys. 07 (2001) 045 [hep-ph/0105212];
W. Grimus and L. Lavoura, *Softly broken lepton numbers: an approach to maximal neutrino mixing*, Acta Phys. Polon. B 32 (2001) 3719 [hep-ph/0110041].

[5] W. Grimus and L. Lavoura, *A discrete symmetry group for maximal atmospheric neutrino mixing*, Phys. Lett. B 572 (2003) 189 [hep-ph/0305046 v2].

[6] W. Grimus and L. Lavoura, *A non-standard CP transformation leading to maximal atmospheric neutrino mixing*, to be published in Phys. Lett. B [hep-ph/0305309 v3].

[7] P.F. Harrison and W.G. Scott, *μ–τ reflection symmetry in lepton mixing and neutrino oscillations*, Phys. Lett. B 547 (2002) 219 [hep-ph/0210197].

[8] G. Ecker, W. Grimus, and W. Konetschny, *Quark mass matrices in left–right symmetric gauge theories*, Nucl. Phys. B 191 (1981) 465;
G. Ecker, W. Grimus, and H. Neufeld, *Spontaneous CP violation in left–right symmetric gauge theories*, Nucl. Phys. B 247 (1984) 70;
W. Grimus and M.N. Rebelo, *Automorphisms in gauge theories and the definition of CP and P*, Phys. Rept. 281 (1997) 239 [hep-ph/9506272].

[9] T. Ohlsson and G. Seidl, *A flavor symmetry model for bilarge leptonic mixing and the lepton masses*, Nucl. Phys. B 643 (2002) 247 [hep-ph/0206087];
G. Seidl, *Deconstruction and bilarge neutrino mixing*, hep-ph/0301044.